PLANE CONTACT PROBLEM OF THE THEORY OF ELASTICITY WITH BONDING OR FRICTIONAL FORCES

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Unlike the results obtained in [1 to 3], the methods given below are based on a certain property of Jacobi polynomials of a particular type.

Proof of this property is given in par. 1. In par. 2 we consider a contact problem for a semi – infinite plane with frictional or bonding forces within the zone of contact and with thermal stresses present. In par. 3 formulas are given for computing the field of stress in the semi-infinite plane under the action of a punch with bonding. In par. 4 we apply the plane contact problem with bonding or frictional forces taken into account, to an elastic foundation of a general type. All the listed contact problems are solved for the case of a single region of contact, and plane deformation is assumed everywhere.

1. In this paragraph we shall prove the following important relationship

$$\int_{-1}^{1}\left[\frac{\operatorname{sgn}(\xi-\tau)}{2}+\frac{\cot\pi\gamma}{\pi}\ln\frac{1}{|\xi-\tau|}\right]\frac{P_m^{\gamma}(\tau)}{\varphi_\gamma(\tau)}\,d\tau=\mu_m P_m^{-\gamma}(\xi)\qquad\left(\begin{matrix}|\xi|\leqslant 1,\ |\operatorname{Re}\gamma|<{}^1/_2\\ m=0,1,2,\ldots\end{matrix}\right)\tag{1.1}$$

where

$$2\mu_0=\pi\sec\pi\gamma-2\operatorname{cosec}\pi\gamma\,[\ln 2+\psi(0.5+\gamma)-\psi(1)],\qquad \mu_m=(m\sin\pi\gamma)^{-1}\quad (m=1,2\ldots)\tag{1.2}$$

$$P_m^{\gamma}(x)=P_m^{\gamma-1/2,\,-\gamma-1/2}(x),\qquad \varphi_\gamma(x)=(1-x)^{-\gamma+1/2}(1+x)^{\gamma+1/2}$$

Here $P_m^{\alpha,\beta}(x)$ is a Jacobi polynomial and $\psi(x)$ is Euler's *psi*-function [4]. The relationship (1.1) is proved in [5] for the case $m=0$, hence only the case $m\geqslant 1$ remains to be considered. We shall begin by evaluating the integral

$$L(z)=\int_{-1}^{1}\ln(z-t)\,\frac{P_m^{\alpha,\beta}(t)\,dt}{(1-t)^{-\alpha}(1+t)^{-\beta}}\qquad[\operatorname{Re}(\alpha,\beta)>-1,\ z\,\overline{\in}\,(-1,1)]\tag{1.3}$$

Using the formula [4]

$$P_m^{\alpha,\beta}(t) = \frac{(-1)^m}{2^m m!}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{d^m}{dt^m}[(1-t)^{\alpha+m}(1+t)^{\beta+m}] \qquad (1.4)$$

and integrating by parts, we have

$$L(z) = -\frac{1}{2^m m}\int_{-1}^{1}\frac{(1-t)^{\alpha+m}(1+t)^{\beta+m}}{(z-t)^m}\,dt$$

Expanding the denominator in the integrand into a series in increasing powers of $(1-t)$ and subsequently integrating term by term, we obtain

$$L(z) = \frac{(-1)^{m+1}2^{1+\alpha+\beta+m}\Gamma(1+\alpha+m)\Gamma(1+\beta+m)}{m(1-z)^m\Gamma(2+\alpha+\beta+2m)}\,{}_2F_1\left(\begin{matrix}m,1+\alpha+m;\\ \alpha+\beta+2m+2;\end{matrix}\ \frac{2}{1-z}\right) \qquad (1.5)$$

Let now z tend to the points on the segment $(1\ -1,\ 1)$ of the real axis, assuming at the same time that

$$\begin{gathered}\ln(z-s)\to\ln|x-s|, \qquad z\to x\pm i0,\ x>s\\ \ln(z-s)\to\ln|x-s|\pm i\pi \quad z\to x\pm i0,\ x<s\end{gathered} \qquad (1.6)$$

and remembering that the hypergeometric function appearing in (1.5) is multivalued.

To obtain the single-valued branch, we must introduce a cut along the $(-1,\ 1)$ segment of the real axis. Values of this function on the edges of this cut are easily found if use is made of the formula ([6], p. 111)

$$\begin{aligned}&{}_2F_1\left(\begin{matrix}m,1+\alpha+m;\\ \alpha+\beta+2m+2;\end{matrix}\ \frac{2}{1-(x\pm i0)}\right) =\\ &=\left(\frac{1-x}{2}\right)^m\left\{\frac{(\alpha+\beta+m+2)_m}{(1+\alpha)_m(-1)^m}\,{}_2F_1\left(\begin{matrix}m,-1-\alpha-\beta-m;\\ -\alpha;\end{matrix}\ \frac{1-x}{2}\right)+\right.\\ &+\frac{\Gamma(\alpha+\beta+2m+2)\Gamma(-1-\alpha)}{\Gamma(m)\Gamma(1+\beta+m)(-1)^m}\left(e^{\mp i\pi}\frac{2}{1-x}\right)^{-1-\alpha}{}_2F_1\left(\begin{matrix}-\beta-m,1+\alpha+m;\\ 2+\alpha;\end{matrix}\ \frac{1-x}{2}\right)\end{aligned} \qquad (1.7)$$

Formula (1.3) can be written as

$$L(z) = \left(\int_{-1}^{x}+\int_{x}^{1}\right)\ln(z-t)(1-t)^{\alpha}(1+t)^{\beta}P_m^{\alpha,\beta}(t)\,dt \qquad (1.8)$$

from which, together with (1.6), we easily obtain

$$\frac{1}{2}[L(x+i0)+L(x-i0)] = \int_{-1}^{1}\ln|x-t|(1-t)^{\alpha}(1+t)^{\beta}P_m^{\alpha,\beta}(t)\,dt$$

On the other hand, (1.5) and (1.7) yield

$$\frac{1}{2}[L(x+i0)+L(x-i0)] =$$

$$= \frac{\Gamma(-1-\alpha)\Gamma(1+\alpha+m)\cos\pi\alpha}{2^{-\beta} m!\,(1-x)^{-1-\alpha}}\,{}_2F_1\left(\begin{matrix}\beta-m,\ 1+\alpha+m;\ \frac{1-x}{2}\\ 2+\alpha;\end{matrix}\right) -$$

$$-\frac{2^{1+\alpha+\beta}\Gamma(1+\alpha)\Gamma(1+\beta+m)}{m\Gamma(a+\beta+m+2)}\,{}_2F_1\left(\begin{matrix}m,\ -1-\alpha-\beta-m;\ \frac{1-x}{2}\\ -\alpha;\end{matrix}\right)$$

hence

$$\int_{-1}^{1} \ln|x-t|\,(1-t)^{\alpha}(1+t)^{\beta} P_m^{\alpha,\beta}(t)\,dt =$$

$$= -\frac{2^{1+\alpha+\beta}\Gamma(1+\alpha)\Gamma(1+\beta+m)}{m\Gamma(\alpha+\beta+m+2)}\,{}_2F_1\left(\begin{matrix}m,\ -1-\alpha-\beta-m;\ \frac{1-x}{2}\\ -\alpha;\end{matrix}\right) + \tag{1.9}$$

$$+\frac{\Gamma(1+\alpha+m)\cot\pi\alpha}{2^{-\beta} m!\,(1-x)^{-1-\alpha}}\,{}_2F_1\left(\begin{matrix}\beta-m,\ 1+\alpha+m;\ \frac{1-x}{2}\\ 2+\alpha;\end{matrix}\right)\frac{\pi}{\Gamma(2+\alpha)}$$

Taking into account (1.4) we can easily confirm that

$$\int_{-1}^{1} \frac{1}{2}\operatorname{sign}(x-t)\,(1-t)^{\alpha}(1+t)^{\beta} P_m^{\alpha,\beta}(t)\,dt =$$

$$= -(2m)^{-1}(1-x)^{1+\alpha}(1+x)^{1+\beta} P_{m-1}^{\alpha+1,\,\beta+1}(x)$$

If we however assume that [4]

$$n!\,P_n^{\alpha,\beta}(x) = (1+\alpha)_n\,{}_2F_1(1+\alpha+\beta+n,\ -n;\ 1+\alpha;\ {}^1/_2(1-x))$$
$${}_2F_1(\alpha,\beta;\ \gamma;\ z) = (1-z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z) \tag{1.10}$$

then

$$\int_{-1}^{1} \frac{1}{2}\operatorname{sign}(x-t)\,\frac{P_m^{\alpha,\beta}(t)\,dt}{(1-t)^{-\alpha}(1+t)^{-\beta}} =$$

$$= -\frac{2^{\beta}\Gamma(1+\alpha+m)}{m!\,\Gamma(2+\alpha)}(1-x)^{1+\alpha}\,{}_2F_1\left(\begin{matrix}-\beta-m;\ 1+\alpha+m;\ \frac{1-x}{2}\\ 2+\alpha;\end{matrix}\right)$$

Finally, adding to the above (1.9) multiplied by $\pi^{-1}\tan\pi\alpha$, we shall find that

$$\int_{-1}^{1}\left[\frac{1}{2}\operatorname{sign}(x-t)-\frac{\operatorname{tg}\pi\alpha}{\pi}\ln\frac{1}{|x-t|}\right]\frac{P_m^{\alpha,\beta}(t)\,dt}{(1-t)^{-\alpha}(1+t)^{-\beta}} =$$

$$= -\frac{\tan\pi\alpha\Gamma(1+\alpha)\Gamma(1+\beta+m)}{2^{-1-\alpha-\beta}\pi m\Gamma(\alpha+\beta+m+2)}\,{}_2F_1\left(\begin{matrix}m,\ -1-\alpha-\beta-m;\ \frac{1-x}{2}\\ -\alpha\end{matrix}\right)$$

from which, assuming that $\alpha = \gamma - {}^1/_2$, $\beta = -\gamma - {}^1/_2$ and taking (1.10) into account, we obtain (1.1).

2. If surface stresses (here $p(x)$ are the normal, while $q(x)$ are the tangential stresses) and the temperature $t(x, 0) = g_1(x)$, both the stresses and the temperature tending to

zero at infinity, are given on the boundary of the semi-infinite plane ($-\infty < x < \infty$, $0 < y < \infty$) in a stationary temperature field, the methods of operational calculus [7] can be used to obtain the following formulas for the vertical (parallel to y-axis) $u_1(x)$ and horizontal $u_2(x)$ displacements of the points on the boundary ($y = 0$) of the semi-infinite plane:

$$u_j(x) + C_j = \theta_2 \left[\sum_{k=1}^{2} \int_{-\infty}^{\infty} K_{jk}(x-s)\, p_k(s)\, ds + \frac{\delta_{j2}\sigma}{1-2\nu} \int_{-\infty}^{\infty} K_{1j}(x-s)\, g_1(s)\, ds \right] \qquad (j = 1, 2)$$

$$p_1 = p - \sigma g_1, \quad p_2 = q; \qquad \theta_1\sigma = (1+\nu)\alpha, \qquad E\theta_1 = 2(1-\nu^2) \qquad (2.1)$$

$$E\theta_2 = (1+\nu)(1-2\nu)$$

$$K_{11} = K_{22} = -\frac{\varkappa \ln|x|}{\pi}, \qquad K_{12} = -K_{21}(x) = \frac{\operatorname{sgn} x}{2}, \qquad \varkappa = \frac{\theta_1}{\theta_2} = \frac{2(1-\nu)}{1-2\nu}$$

Here δ_{jk} is the Kronecher symbol; E, ν and α are the modulus of elasticity, Poisson coefficient, and the coefficient of linear expansion respectively, while C_1 and C_2 are undefined constants. With the formulas (2.1) available, it is easy to formulate a number of contact problems with the thermal stresses taken into account.

Let us for example consider a punch with a plane rectilinear face (length $2a$) in contact with an elastic semi-plane and acted upon by an arbitrary system of forces. We shall assume here that the temperature of the face of the punch is given by the function $g_1(x)$, $-a \leqslant x \leqslant a$ while that of the part of the boundary of the semi-plane not in contact with the punch is given by $t(x, 0) = g_2(x)$, $|x| > a$.

We shall also assume the contact between the punch and the semi-plane is complete [7] and that $t(x, 0) = g_1(x)$, $|x| \leqslant a$.

Then, by (2.1), contact stresses $p(x)$ and $g(x)$ should satisfy the following system of integral equations

$$\sum_{k=1}^{2} \int_{-a}^{a} K_{jk}(x-s)\, p_k(s)\, ds = \sigma \Big[\delta_{j1} \int_{l} K_{j1}(x-s)\, g_2(s)\, ds +$$

$$+ \frac{\delta_{j2}}{1-2\nu} \int_{-a}^{a} K_{j1}(x-s)\, g_1(s)\, ds + \delta_{j1}\theta_2^{-1}\Theta x + C_j \qquad (j = 1, 2, \; -a < x < a)$$

Here Θ is the angle of inclination of the stamp, l is the real axis with the interval $(-a, a)$ excluded, C_j are arbitrary constants, differing however from those in (2.1).

Let us now make the substitution $x = a\xi$, $s = a\tau$, multiply the first of the obtained equations ($j = 1$) by i and substract from it the second one ($j = 2$) to obtain a single integral equation

$$\int_{-1}^{1} \left[\frac{1}{2} \operatorname{sign}(\xi - \tau) + \frac{\cot \pi\gamma}{\pi} \ln \frac{1}{|\xi - \tau|} \right] \chi(\tau)\, d\tau = f(\xi) \quad (-1 < \xi < 1) \qquad (2.2)$$

for the function

$$\chi(\xi) = ap(a\xi) - \sigma a g_1(a\xi) + iaq(a\xi) \qquad (2.3)$$

Here we have assumed, that

$$i\varkappa = \cot \pi\gamma; \ \gamma = -i\mu, \ 2\pi\mu = \ln\{[(\varkappa + 1)/(\varkappa - 1)] = 3 - 4\nu\} \coth \pi\mu = \varkappa \tag{2.4}$$

The right-hand side of (2.2) is of the type

$$f(\xi) = \sigma\left[\int_{-1}^{1} \frac{\operatorname{sign}(\xi - \tau)}{2(1-2\nu)} a g_1(a\tau)\, d\tau + \frac{i\varkappa\sigma}{\pi}\int_{l'} \ln\frac{1}{|\xi - \tau|}\, a g_2(a\tau)\, d\tau\right] + \\ + iC_1 - C_2 + i\omega\xi \qquad (-1 \leqslant \xi \leqslant 1, \quad \omega = a\Theta\theta_2^{-1}) \tag{2.5}$$

where l' denotes the real axis without the interval (−1, 1).

If on the other hand the contact between the punch and the semi-plane gives rise to frictional instead of bonding forces and a critical state [1, 2 and 3] $q(x) = kp(x)$ is reached, then, using the first ($j = 1$) formula of (2.1) as a starting point, we can reduce this problem to (2.2). This time however, unlike the previous case, we shall have

$$\chi(\xi) = ap(a\xi) - \sigma a q_1(a\xi), \qquad \pi\gamma = \cot^{-1}(\varkappa / k), \qquad k\omega^* = \omega$$

$$f(\xi) = \sigma\left[\frac{\varkappa}{k\pi}\int_{l'} \ln\frac{1}{|\xi - \tau|}\, a g_2(a\tau)\, d\tau - \int_{-1}^{1} \frac{\operatorname{sign}(\xi - \tau)}{2} a g_1(a\tau)\, d\tau\right] + \omega^*\xi + C_1 \tag{2.6}$$

At present two methods of exact solution of (2.2) are known. In the first one, both parts of the equation are transformed by formal differentiation into an ordinary singular integral equation which is solvable in the real form. This method was employed by Shtaerman [2] *.

The other method is due to Krein. In it, the limits of integration must first be changed to $(-a, a)$, then a formula applied (formulas (3.4) and (3.5) in [5]) leading to the solution in which the right-hand side of the equation becomes identically equal to zero, and finally, use made of the Krein's result.

Both methods give the solution in form of quadratures. These however become complex for more compilcated right-hand sides, since in the first method the solution is given in form of an integral the principal value of which must be used, while the second method uses double integrals of appreciable complexity.

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* Obviously, the appropriate problems are reduced to the integral equation (2.2) without the thermal stresses. In the course of solving a plane contact problem of the theory of elasticity with the frictional forces, Shtaerman [2] obtained an equation, differing slightly in structure from (2.2). Use of an additive constant makes it possible to transform this equation into (2.2), but then the coefficient of the logarithm will be different. The reason for this is that Shtaerman neglected, in his derivation (this was pointed out by Ia. L. Nudel'man in 1955), the second term in the expression $\tau_{xy} = G(\partial v / \partial x + \partial u / \partial y)$. To make Shtaerman's results correct (and in line with the formulas of Muskhelishvili [1]), it is necessary to compute the coefficient ν appearing in his formulas, according to

$$\nu = (\theta_1^{(1)} + \theta_1^{(2)})k^{-1}(\theta_2^{(1)} + \theta_2^{(2)}), \ \theta_1^{(j)} = 2(1 - \nu_j^2)(\pi E_j)^{-1}, \ \theta_2^{(j)} = \\ = E_j^{-1}(1 + \nu_j)(1 - 2\nu_j) \qquad (j = 1, 2)$$

where k is the coefficient of friction, ν_j and E_j are the Poisson coefficient and the

(continued on the next page)

The relation obtained in the previous paragraph leads to yet another method of solution of (2.2), and we shall utilise it in the present work.

In order to obtain the solution of (2.2) by this method we must, in general, expand the right hand-side into a series

$$f(\xi) = \sum_{m=0}^{\infty} A_m P_m^{-\gamma}(\xi) \tag{2.7}$$

Then, by (1.1), the solution will be of the type

$$\chi(\xi) = \sum_{m=0}^{\infty} \frac{A_m}{\mu_m} \frac{P_m^{\gamma}(\xi)}{\varphi_\gamma(\xi)}$$

In many problems, the right-hand side either is a polynomial, or can be well approximated by a polynomial. In such cases, we must obtain a solution $\chi_n(\xi)$ of (2.2) the right-hand side of which, is

$$f(\xi) = \xi^n = \sum_{m=0}^{n} C_m^n(-\gamma) P_m^{-\gamma}(\xi) \tag{2.8}$$

where

$$C_m^n(\gamma) = \frac{1}{\lambda_m} \int_{-1}^{1} \frac{\xi^n P_m^{\gamma}(\xi)}{\varphi_\gamma(\xi)} d\xi = \frac{1}{\lambda_m} \sum_{j=m}^{n} (-1)^j \binom{n}{j} b_j^m(\gamma) \tag{2.9}$$

$$\lambda_m = \int_{-1}^{1} \frac{[P_m^{\gamma}(\xi)]^2}{\varphi_\gamma(\xi)} d\xi = \begin{cases} \pi \sec \pi\gamma, & m = 0 \\ (m!^2 2)^{-1} \Gamma(m + \gamma + 1/2)\, \Gamma(m - \gamma + 1/2), & m = 1, 2, \ldots \end{cases} \tag{2.10}$$

$$b_j^m(\gamma) = \int_{-1}^{1} \frac{P_m^{\gamma}(\xi)(1-\xi)^j d\xi}{\varphi_\gamma(\xi)} = \begin{cases} 0, & j < m \\ \dfrac{2^j j!\, \Gamma(m - \gamma + 1/2)\, \Gamma(j + \gamma + 1/2)}{(-1)^m m!\,(j-m)!\,(j+m)!}, & j \geqslant m \end{cases} \tag{2.11}$$

Here we have utilised the formulas 7.391 from [4]. According to (1.1), we have

$$\chi_n(\xi) = \sum_{m=0}^{n} \frac{C_m^n(-\gamma)}{\mu_m} \frac{P_m^{\gamma}(\xi)}{\varphi_\gamma(\xi)} \tag{2.12}$$

To illustrate the application of the obtained formulas, we shall use the following problem. Let a punch with a plane face (width $2a$) be pressed into an elastic semi-infinite plane acted upon by an arbitrary system of forces (P and Q being vertical and horizontal components of the principal vector respectively, and M the principal moment). We assume that coupling exists between the punch and the face and, that the boundary of the semi-plane not under the punch, is maintained at zero temperature. The functions defining the temperature of the face of the punch shall be approximated by means of a polynomial, i.e.

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modulus of elasticity, respectively, for the first ($j = 1$) and the second ($j = 2$) body.

$$g_1(x) = \sum_{j=0}^{n-1} A_j^* x^j \qquad (g_2(x) \equiv 0) \tag{2.13}$$

In this case, according to (2.5), the right-hand side of (2.2) will assume the form

$$f(\xi) = iC_1 - C_2 + i\omega\xi + \frac{\sigma}{1-2\nu}\sum_{j=1}^{n} \frac{A_{j-1}^* a^j}{j}\xi^j \tag{2.14}$$

Using (2.12) we shall easily find the solution of (2.2) corresponding to the right-hand side of (2.13)

$$\chi(\xi) = \frac{C + 2i\omega \sin \pi\gamma P_1^{\gamma}(\xi)}{\varphi_\gamma(\xi)} + \sum_{m=1}^{n} \frac{\sigma m A_m^n P_m^{\gamma}(\xi)}{(1-2\nu)\operatorname{cosec}\pi\gamma\varphi_\gamma(\xi)}$$

$$A_m^n = \sum_{j=m}^{n} \frac{A_{j-1}^* C_m^j(-\gamma)}{ja^{-j}} \tag{2.15}$$

where C is an arbitrary constant to be determined later.

Conditions of equilibrium of the punch with (2.3) taken into account, can be written as

$$P + iQ - \sigma\sum_{j=1}^{n} A_{j-1}^* \frac{1-(-1)^j}{j} a^j = \int_{-1}^{1} \chi(\xi)\, d\xi$$

$$\frac{M}{a} - \sigma\sum_{j=1}^{n} A_{j-1}^* \frac{1+(-1)^j}{j+1} a^j = \operatorname{Re}\int_{-0}^{1} \chi(\xi)\,\xi d\xi \tag{2.16}$$

Let us now put the expression (2.15) under the integral sign of the first formula of (2.16). Orthogonality of Jacobi polynomials and use of (2.10) and (2.4) lead, on integration, to

$$C = \frac{\cosh \pi\mu}{\pi}\left(P + iQ - 2\sigma\sum_{j=1}^{(n+1)} \frac{a^{2j-1} A_{2j-2}^*}{2j-1}\right) \tag{2.17}$$

Here and in the following, (n) will denote either a whole number $\frac{1}{2}n$ or the nearest smaller whole number. In order to utilise the second condition of equilibrium, we shall have to separate the real part of $\chi(\xi)$. In connection with this we shall note that the values given by (2.10) are real even when $\gamma = -i\mu$ ($\operatorname{Im}\mu = 0$).

From the integral representation (2.9) for $C_m^n(\gamma)$ we can, by replacing the variable of integration ξ with $-\xi$ and utilising a known ([4], p. 1049) relationship

$$P_n^{\alpha,\beta}(-x) = (-1)^n P_n^{\beta,\alpha}(x) \tag{2.18}$$

establish, that

$$C_m^n(\gamma) = (-1)^{n+m}\overline{C_m^n(\gamma)} \qquad (\operatorname{Re}\gamma = 0,\ C_m^m = m!\,2\,[(2m-1)!!]^{-1},\ m \geqslant 1,\ C_0^0 = 1)$$

Here the bar denotes a complex conjugate. The formula in brackets follows from the second equality of (2.9).

From the second formula of (2.15) we have

$$A_{2k}^{n} = \sum_{j=k}^{(n)} C_{2k}^{2j} a^{2j} \frac{A_{2j}^{*} - 1}{2j} + i \sum_{j=k}^{(n-1)} C_{2k}^{[2j+1]} a^{2j+1} \frac{A_{2j}^{*}}{2j+1} \qquad (k = 1, 2, \ldots)$$
$$A_{2k-1}^{n} = \sum_{j=k}^{(n+1)} C_{2k-1}^{2j-1} a^{2j-1} \frac{A_{2j-2}^{*}}{2j-1} + i \sum_{j=k}^{(n)} C_{2k-1}^{[2j]} a_{2j} \frac{A_{2j-1}^{*}}{2j} \tag{2.19}$$

where

$$C_m^{[n]} = -iC_m^{n}$$

Let us now put (2.15) under the integral sign of the second formula of (2.16) and let us integrate it, using (2.9). Separating the real part by means of (2.19), we shall obtain the formula

$$\omega = \frac{\varkappa}{2\pi(0.25+\mu^2)} \left\{ \frac{M}{a} + 2\mu Q - \sigma \sum_{j=1}^{(n)} \frac{A_{2j-1}}{a^{-2j}} \left[\frac{2}{2j+1} + \frac{\pi(0.25+\mu^2) C_1^{[2j]}}{\varkappa(1-2\nu)2j} \right] \right\} = \frac{\Theta a}{\theta_2} \tag{2.20}$$

for the angle Θ of rotation of the punch.

Formulas (2.3), (2.15), (2.17) and (2.20) give a complete solution of the problem. When separating the real from the imaginary part in (2.15), we must remember that, when $\gamma = -i\mu$ ($\operatorname{Im} \mu = 0$), then by (1.4), we have

$$\frac{P_m^{\gamma}(\xi)}{\varphi_\gamma(\xi)} = \frac{(-1)^m}{2^m m!} \left\{ \frac{d^m}{d\xi^m} \left[(1-\xi^2)^{m-1/2} \cos \mu \ln \frac{1+\xi}{1-\xi} \right] + \right.$$
$$\left. + i \frac{d^m}{d\xi^m} \left[(1-\xi^2)^{m-1/2} \sin \mu \ln \frac{1+\xi}{1-\xi} \right] \right\} \tag{2.21}$$

Let now the temperature of the face of the punch be constant, i.e.

$$g_1(x) = t_0 \qquad (A_0^* = t_0, \ A_j^* = 0, \ j = 1, 2 \ldots) \tag{2.22}$$

In this case by (2.3), (2.15), (2.17) and (2.20), we have, for the normal contact stress,

$$p(x) = \sigma t_0 + \frac{\varkappa(\varkappa^2-1)^{-1/2}}{\pi\sqrt{a^2-x^2}} \left\{ \left[P + \frac{2\mu Q + a^{-1}M}{2(0.25+\mu^2)} \frac{x}{a} - 2a\sigma t_0 \left(1 + \frac{\mu\pi}{\varkappa(1-2\nu)} \right) \right] \times \right.$$
$$\left. \times \cos \mu \ln \frac{a+x}{a-x} + \left[\frac{(\mu^2 - 0.25) Q + \mu a^{-1} M}{\mu^2 + 0.25} + \frac{x\sigma\pi t_0}{(1-2\nu)\varkappa} \right] \sin \mu \ln \frac{a+x}{a-x} \right\}$$

For the tangential contact stress the formula will be analogous (without however the first term). It shall not be quoted here. (2.20) in which the term containing the summation is neglected, gives the corresponding value for the angle of rotation of the punch. If $t_0 = M = Q = 0$, then the last expression becomes a formula given by Muskhelishvili ([1], p. 433), provided it is previously divided by 2 (a misprint).

Let us now consider the case (2.13) in which frictional forces replace the bonding forces. In this case the right-hand side of (2.2) becomes, by (2.6),

$$f(\xi) = C_1 + \omega^* \xi - \sigma \sum_{j=1}^{n} \frac{A_{j-1}^{*}}{j} a^j \xi^j$$

Use of (2.12) enables us to obtain the solution of (2.2) in the form

$$\chi(\xi) = \frac{C + 2\omega^* \sin\pi\gamma P_1^{\gamma}(\xi)}{\varphi_\gamma(\xi)} - \sigma \sum_{m=1}^{n} \frac{m A_m^{\ n} \sin\pi\gamma}{\varphi_\gamma(\xi)} P_m^{\gamma}(\xi) \tag{2.23}$$

Here $A_m^{\ n}$ is defined by the second formula of (2.15), and γ by the corresponding formula of (2.6).

Conditions of equilibrium of the punch lead us again, as in the case of bonding forces, to

$$\omega^* = \frac{\cot\pi\gamma}{2\pi(0.25-\gamma^2)}\left[\frac{M}{a} + \frac{2\pi\gamma C}{\cos\pi\gamma} - 2\sigma\sum_{j=1}^{(n)} \frac{A_{2j-1}^{*}a^{2j}}{2j+1}\right] + \frac{\sigma}{2}\sum_{j=1}^{n} \frac{A_{j-1}^{*}}{ja^{-j}} C_1^{\,j}(-\gamma)$$

$$C = \frac{\cos\pi\gamma}{\pi}\left[P - 2\sigma\sum_{j=1}^{(n+1)} \frac{A_{2j-2}^{*}}{j} a^{2j-1}\right] \tag{2.24}$$

Formulas (2.6), (2.23) and (2.24) allow us to solve the considered contact problem in the presence of frictional forces, with the condition (2.13) in force. When the temperature of the punch is constant i.e. (2.22), the formulas for the normal contact stress and the angle of rotation of the punch, become

$$p(x) = \frac{\cos\pi\gamma\,(a^2-x^2)^{\gamma-1/2}}{\pi(0.25-\gamma^2)(a^2+x^2)^{\gamma+1/2}}\left[(P-2a\sigma t_0)\left(0.25+\gamma^2+\frac{\gamma x}{a}\right) + \right.$$

$$\left. + \frac{M}{a}\left(\gamma + \frac{x}{2a}\right) - a\pi\sigma t_0 \sec\pi\gamma\,(1-\sin\pi\gamma)(0.25-\gamma^2)(2\gamma + a^{-1}x)\right] + \sigma t_0$$

$$\omega^* = \frac{a\Theta}{k\theta_2} = \frac{\cot\pi\gamma}{2\pi(0.25-\gamma^2)}\left[\frac{M}{a} + 2\gamma P - 4a\gamma\sigma t_0\right] + \sigma t_0$$

Putting $t_0 = 0$, results in Muskhelishvili formulas ([1], p. 452).

3. The solutions of contact problems obtained in the last paragraph in the form of Jacobi polynomials, are suitable for computing the stress and strain fields inside the semi-plane. We shall illustrate the point with pure (without thermal stresses) elastic problems.

In the general case of the problem with bonding forces, contact stresses $p(x)$ and $q(x)$ will, in view of the previous argumenst, be given by

$$\chi(\xi) = ap(a\xi) + iaq(a\xi) = \sum_m \frac{A_m}{\mu_m}\chi_m^*(\xi), \qquad \chi_m^*(\xi) = \frac{P_m^{\gamma}(\xi)}{\varphi_\gamma(\xi)} \tag{3.1}$$

According to [1], the stress field in the semi-plane can be found by utilising the relation

$$\sigma_x + \sigma_y = 2\,[\Phi(z) + \overline{\Phi(z)}],\quad \sigma_y - i\tau_{xy} = \Phi(z) - \Phi(\bar z) + (z - \bar z)\overline{\Phi'(z)} \tag{3.2}$$

where in our case

$$\Phi(z) = \frac{1}{a}\Psi\left(\frac{z}{a}\right), \qquad \Psi(\zeta) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\chi(\xi)\,d\xi}{\xi-\zeta} \tag{3.3}$$

Using (3.1), we can write

$$\Psi(\zeta) = \sum_m \frac{A_m}{\mu_m}\Psi_m(\zeta), \qquad \Psi_m(\zeta) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{P_m^{\gamma}(\xi)}{\varphi_\gamma(\xi)}\frac{d\xi}{\xi-\zeta} \tag{3.4}$$

But the last integral represents, with the accuracy of up to the value of the multiplier, a Jacobi function of the second kind, which can be expressed in terms of Gauss' hypergeometric function ([9], p. 86), as

$$\Psi_m(\zeta) = \frac{2^{m-1}\Gamma(m+\gamma+{}^1/_2)\,\Gamma(m-\gamma+{}^1/_2)\,i}{(2m)!\,\pi\,(\zeta-1)^{m+1}}\;{}_2F_1\begin{pmatrix} m+\gamma+{}^1/_2, & m+1; & \dfrac{2}{1-\zeta} \\ 2m+1; & & \end{pmatrix} \quad (3.5)$$

Further, by the formula 9.132 (1) of [4], we have

$$\begin{gathered}{}_2F_1\begin{pmatrix} m+\gamma+{}^1/_2, m+1; & \dfrac{2}{1-\gamma} \\ 2m+1; & \end{pmatrix} = \frac{(2m)!\;(m!)^{-1}}{(-\gamma+{}^1/_2)_m}\left(\frac{\zeta-1}{\zeta+1}\right)^{m+\gamma+{}^1/_2}{}_2F_1\begin{pmatrix} m, & m+\gamma+{}^1/_2; & \dfrac{\zeta-1}{\zeta+1} \\ & \gamma+{}^1/_2; & \end{pmatrix} + \\ + \frac{(2m)!}{(m-1)!(\gamma-{}^1/_2)_{m+1}}\,\frac{(\zeta-1)^{m+1}}{(\zeta+1)^{m+1}}\;{}_2F_1\begin{pmatrix} m+1, & m-\gamma+{}^1/_2; & \dfrac{\zeta-1}{\zeta+1} \\ & -\gamma+{}^3/_2; & \end{pmatrix}\end{gathered} \quad (3.6)$$

which can be further simplified with the help of

$${}_2F_1\begin{pmatrix} a, & b+m; \\ b; & x \end{pmatrix} = \sum_{j=0}^{\infty}\frac{(a)_j\,(b+m)_j}{j!\,(b)_j}\,x^j = \frac{m!}{(1-x)^a}\sum_{k=0}^{m}\frac{(a)_k x^k\,(1-x)^{-k}}{k!\,(m-k)!\,(b)_k} \quad (3.7)$$

To confirm that the second equality is true, we must replace the term in the infinite sum, with that given in [6] p. 7.

$$\frac{(b+m)_j}{(b)_j} = \sum_{k=0}^{j}\binom{j}{k}\frac{(m+1-k)_k}{(b)_k} = m!\sum_{k=0}^{\infty}\frac{j!}{\Gamma(j-k+1)\,\Gamma(m-k+1)\,k!\,(b)_k}$$

and change the order of summation.

Use of (3.7) to transform the right hand side of (3.6) results in

$$\begin{gathered}\Psi_m(\zeta) = \frac{i\Gamma(m+\gamma+{}^1/_2)}{2\pi\varphi_\gamma(\zeta)}\sum_{k=0}^{m}\frac{(m)_k\,\Gamma(-\gamma+{}^1/_2)\,(\zeta-1)^k}{k!\,(m-k)!\,(\gamma+{}^1/_2)_k\,2^k} - \\ -\frac{\Gamma(\gamma-{}^1/_2)}{4\pi i}\sum_{k=0}^{m-1}\frac{(m+1)_k\,\Gamma(m-\gamma+{}^1/_2)\,(\zeta-1)^k}{k!\,(m-1-k)!(-\gamma+{}^3/_2)_k\,2^k}\end{gathered} \quad (3.8)$$

in place of (3.5). The second sum in the above expression should be omitted when $m = 0$.

So we arrive at the set (3.2) to (3.4) which, together with (3.8), define in terms of elementary functions, the stress field in the semi-plane in case of the contact problem with the bonding forces present. The parameter γ is defined here by (2.4).

For the contact problem with frictional forces we have, in general,

$$ap(a\xi) = \sum_m \frac{A_m}{\mu_m}\,\frac{P_m^{\,\gamma}(\xi)}{\varphi_\gamma(\xi)}, \qquad q(x) = kp(x)$$

The field of stress will be given here by (3.2) and (3.3), and

$$\Psi(\zeta) = (1+ik)\sum_m \frac{A_m}{\mu_m}\Psi_m(\zeta)$$

where the formula (3.8) is valid for $\Psi_m(\zeta)$, while the second formula of (2.6) is valid for γ.

From (3.8) we obtain, when $\gamma = 0$, the solution of the contact problem with tangential contact stresses absent.

4. Consider an elastic foundation of a general type. Following [5] we shall assume that the influence functions are given by

$$v_m^*(x) = \theta_1 v_m\left(\frac{x}{h}\right) \qquad (m = 0, 2), \qquad v_1^*(x) = \theta_2 v_1\left(\frac{x}{h}\right)$$

$$v_{0,2}(s) = \frac{1}{\pi}\int_0^\infty \frac{\psi_{0,2}(t)\cos ts}{t}\,dt \tag{4.1}$$

$$v_1(s) = \frac{1}{\pi}\int_0^\infty \frac{\psi_1(t)}{t}\sin ts\,dt, \quad \psi_{0,2}(0) = 0; \quad \psi_m(t) = 1 + o(1), \quad t \to \infty \ (m = 0, 1, 2)$$

where $v_0^*(x)$ defines the vertical displacement of the points of the surface of the foundation under the action of the unit vertical force applied at the point ($x = 0$, $y = 0$) of the surface of the foundation, $v_2^*(x)$ are the horizontal displacements due to a horizontal force, $v_1^*(x)$ are the vertical displacements caused by horizontal forces or vice-versa; θ_1, θ_2 and h are some positive parameters, and $\theta_1 > \theta_2$. [5] gives the expressions for $\psi_m(t)$ ($m = 0, 1, 2$) for the foundation in the form of a layer with one fixed boundary (in this case h is the thickness of the layer, while θ_1 and θ_2 are given by the formulas from (2.1)).

If we denote the length of the area of contact by $2a$, normal contact stress by $p(x)$, tangential stress by $q(x)$ and vertical and horizontal displacements of the points of the surface within the area of contact by $f_1(x)$ and $f_2(x)$ respectively, then the considered contact problem (without thermal stresses) can easily be reduced to a system of integral equations

$$\varkappa\int_{-1}^{1} v_0\left(\frac{\xi-\tau}{\lambda}\right) ap(a\tau)\,d\tau + \int_{-1}^{1} v_1\left(\frac{\xi-\tau}{\lambda}\right) aq(a\tau)\,d\tau = \frac{f_1(a\xi)}{\theta_2} \qquad \left(\varkappa = \frac{\theta_1}{\theta_2} > 1\right)$$

$$-\int_{-1}^{1} v_1\left(\frac{\xi-\tau}{\lambda}\right) ap(a\tau)\,d\tau + \varkappa\int_{-1}^{1} v_2\left(\frac{\xi-\tau}{\lambda}\right) aq(a\tau)\,dt = \frac{f_2(a\xi)}{\theta_2} \qquad \left(\lambda = \frac{h}{a}\right) \tag{4.2}$$

To arrive at the approximate solution of the above system we shall, utilising (1.1), represent the functions v_m as [5 and 10]

$$v_m(s) = -\frac{\ln|s|}{\pi} - l_m(s), \; l_m(s) = \frac{1}{\pi}\int_0^\infty \frac{[1-\psi_m(t)]\cos ts - e^{-t}}{t}\,dt$$

$$v_1(s) = (^1/_2)\operatorname{sign} s - l_1(s), \; l_1(s) = \frac{1}{\pi}\int_0^\infty \frac{1-\psi_1(t)}{t}\sin ts\,dt \quad (m = 0, 2) \tag{4.3}$$

Since $\psi_m(s)$ are assumed to be asymptotic, the functions $l_m(s)$ will be continuous on the real axis. Moreover, in case of an elastic layer, they will be analytic inside the circle $|s| < 2$.

From the above it follows that they can, within the interval ($-2 < s < 2$) be approximated by means of parts of the corresponding Taylor series

$$l_m(s) = \sum_{j=0}^{n} a_j^m s^{2j} \quad (m = 0, 2), \qquad l_1(s) = \sum_{j=0}^{n} a_j^1 s^{2j+1} \tag{4.4}$$

At the same time by (4.3), we have

$$a_0^{0,2} = \frac{1}{\pi}\int_0^\infty \frac{1-\psi_{0,2}(t)-e^{-t}}{t}\,dt\,, \quad \begin{matrix} a_j^{0,2} \\ a_j^1 \end{matrix} = \frac{1}{\pi}\int_0^\infty \begin{bmatrix} 1-\psi_{0,2}(t) \\ 1-\psi_1(t) \end{bmatrix} \begin{matrix} t^{2j-1} \\ t^{2j} \end{matrix}\,dt \quad (j = 1, 2\ldots)$$

Whenever s is found outside the interval of convergence or $l_m(s)$ are found not to be analytic, the values of $l_m(s)$ $(m = 0, 1, 2)$ must be tabulated by means of (4.3). With such tables available, the functions $l_m(s)$ can again be approximated by means of (4.4), but in this case the coefficients a_j^m must be determined from the condition of minimum deviations in one sense or another.

Let us substitute $v_m(s)$ $(m = 0, 1, 2)$ from (4.3) into (4.2). After the operations resembling those used to obtain (2.2) we shall, in place of (4.2), obtain (comp. [5])

$$\int_{-1}^{1}\left[\frac{1}{2}\operatorname{sign}(\xi-\tau) + \frac{\cot\pi\gamma}{\pi}\ln\frac{1}{|\xi-\tau|}\right]\chi(\tau) -$$

$$-\int_{-1}^{1}\sum_{k=0}^{N}(\xi-\tau)^k\left[C_k^+\chi(\tau) + C_k^-\overline{\chi(\tau)}\right]d\tau = i\alpha(\xi) \qquad (-1<\xi<1,\ N = 2n+1) \tag{4.5}$$

where

$$\begin{aligned} \chi(\xi) &= ap(a\xi) + iaq(a\xi), \\ \alpha(\xi) &= a\theta_2^{-1}\left[f_1(a\xi) + if_2(a\xi)\right], \end{aligned} \qquad i\varkappa = \cot\pi\gamma, \quad \gamma = -\frac{i}{2\pi}\ln\frac{\varkappa+1}{\varkappa-1} \tag{4.6}$$

Coefficients $C_k^{\pm}$ are given in terms of a_k^m according to

$$2\lambda^{2k}C_{2k}^+ = i\varkappa(a_k^\circ - a_k^2), \quad \lambda^{2k+1}C_{2k+1}^+ = a_k^1; \qquad C_{2k+1} = 0 \qquad (k = 0, 1, 2\ldots)$$

$$2C_0^- = \varkappa(a_0^\circ + a_0^2 + 2\pi^{-1}\ln\lambda), \qquad 2\lambda^{2j}C_{2j}^- = i\varkappa(a_j^\circ + a_j^2) \qquad (j = 1, 2\ldots)$$

We shall assume that the function $\alpha(\xi)$ is expanded into a series in terms of Jacobi polynomials

$$\alpha(\xi) = \sum_{k=0}^{\infty}\alpha_k P_k^{-\gamma}(\xi) \tag{4.7}$$

and we shall seek a solution of (4.5) in the form

$$\chi(\tau) = \sum_{m=0}^{\infty}\frac{z_m P_m^{\gamma}(\tau)}{\varphi_\gamma(\tau)}\,, \qquad \overline{\chi(\tau)} = \sum_{m=0}^{\infty}\frac{\overline{z}_m P_m^{-\gamma}(\tau)}{\varphi_{\gamma^-}(\tau)} \tag{4.8}$$

Here z_m are the complex coefficients to be determined. The validity of the second formula of (4.8) is assured by the fact that, by (4.6), $\operatorname{Re}\gamma = 0$ (or $\varkappa > 1$).

Let us now consider the integrals of the type

$$B_{mk}^{l\pm}=\int_{-1}^{1}\int_{-1}^{1}\frac{(\xi-\tau)^k P_l^{\gamma}(\xi)\,P_m^{\mp\gamma}(\tau)\,d\tau}{\varphi_\gamma(\xi)\,\varphi_{\mp\gamma}(\tau)}=\sum_{j=l}^{k-m}(-1)^j\binom{k}{j}b_j^{\,l}(-\gamma)\,b_{k-j}^{\,m}(\pm\gamma) \tag{4.9}$$

Expanding $(\xi-\tau)^k$ into a series in terms of $(1-\xi)$ and using (2.11) we can confirm the second equality in the above formula.

If we now substitute (4.7) and (4.8) into (4.5) and integrate both sides of the obtained equation with the weight function $P_l^{\gamma}(\xi)\,/\,\varphi_\gamma(\xi)$ over the interval $(-1, 1)$, we shall arrive at the following system of algebraic equations

$$z_l=\frac{1}{\mu_l}\Big[i\alpha_l+\frac{1}{\lambda_l}\sum_{m=0}^{N-l}\Big(z_m\sum_{k=m+l}^{N}C_k^{\,+}B_{mk}^{l+}+\bar z_m\sum_{k=m+l}^{N}C_k^{\,-}B_{mk}^{l-}\Big)\Big]\qquad(l=0,1,\ldots,N) \tag{4.10}$$

for the first $N=2n+1$ coefficients $z_m=x_m+iy_m$, while the remaining ones will be given by

$$z_l=i\alpha_l\,/\,\mu_l,\qquad l>N$$

In (4.9) the separation of the real from the imaginary parts is easy, since

$$\overline{B_{mk}^{l\pm}}=(-1)^{m+k+l}B_{mk}^{l\pm}$$

which follows from (4.8) and (2.18).

As a result of this separation we obtain, as in [5], two independently solvable systems of algebraic equations, one which covers x_{2j}, y_{2j+1} $(j=0,1,\ldots,n)$, while the other covers x_{2j+1}, y_{2j} $(j=0,1,\ldots,n)$. They both differ from the case in [5], in that they possess a triangular matrix of coefficients. Apart from this, the solution of the problem is obtained in exactly the same manner, as in [5].

If Coulomb friction is considered instead of bonding forces, the approximate integral equation (4.5) becomes

$$\int_{-1}^{1}\Big[\frac{1}{2}\,\mathrm{sign}\,(\xi-\tau)+\frac{\cot\pi\gamma}{\pi}\ln\frac{1}{|\xi-\tau|}-\sum_{k=0}^{N}C_k^{*}(\xi-\tau)^k\Big]\chi(\tau)\,d\tau=\frac{f_1(a\xi)}{k\theta_4} \tag{4.11}$$

$$(C_0^{*}=\varkappa^{*}(a_0^{\circ}-\pi^{-1}\ln\lambda),\ \lambda^{2k}C_{2k}^{*}=\varkappa^{*}a_k^{\circ},\ k\geqslant 1;\ \lambda^{2k+1}C_{2k+1}^{*}=a_k^{1})$$
$$(k=0,1,2\ldots;\quad \varkappa^{*}=k^{-1}\varkappa)$$

The parameter γ is defined here by the second formula of (2.6), while the normal contact stress in question is found from the formula $\chi(\xi)=ap\,(a\xi)$.

If we expand the function $f_1(x)$ describing the settlement of the foundation in the region of contact into a series in terms of Jacobi polynomials, and seek $X(\xi)$ in form of a series, i.e.

$$f_1(a\xi)=\sum_{l=0}^{\infty}\beta_l P_l^{-\gamma}(\xi),\qquad \chi(\xi)=\sum_{l=0}^{\infty}x_l\frac{P_l^{\gamma}(\xi)}{\varphi_\gamma(\xi)}$$

then, similarly to the previous case, we shall obtain

$$x_l = \frac{1}{\mu_l}\left[\frac{\beta_l}{k\theta_2} + \frac{1}{\lambda_l}\sum_{m=0}^{N-l} x_m \sum_{k=m+l}^{N} C_k^* B_{mk}^{l+}\right] \quad (l < N), \qquad x_l = \frac{\beta_l}{\mu_l k\theta_2} \quad (l > N) \tag{4.12}$$

If the punch has a plane face, then $f_1(x) = \delta + \Theta x$. In this case $\beta_0 = \delta + 2a\Theta\gamma$, $\beta_1 = 2a\Theta$, $\beta_l = 0$ $(l = 2, 3, 4, \ldots)$, and the contact stress must be sought in the form

$$ap(a\xi) = \chi(\xi) = \sum_{m=0}^{N} (\delta_* X_m^\circ + \Theta_* X_m^1)\frac{P_m^\gamma(\xi)}{\varphi_\gamma(\xi)} \qquad \left(\delta_* = \frac{\delta}{k\theta_2}, \quad \Theta_* = \frac{a\Theta}{k\theta_2}\right)$$

Also, by (4.12) the coefficients X_l^j should be obtained from the following system of algebraic equations

$$X_l = \frac{1}{\mu_l}\left[\beta_l^j + \frac{1}{\lambda_l}\sum_{m=0}^{N-l} X_m^j \sum_{k=m+l}^{N} C_k^* B_{mk}^{l+}\right]\left(l \leqslant N; \begin{matrix}\beta_0^\circ = 1, & \beta_l^\circ = 0, & l \geqslant 1 \\ \beta_0^1 = 2\gamma, & \beta_1^1 = 2, & \beta_l^1 = 0, \; l \geqslant 2\end{matrix}\right)$$

possessing a triangular coefficient matrix. Finally, using the conditions of equilibrium of the punch, we obtain the following relations

$$\pi(\delta_* X_0^\circ + \Theta_* X_0^1) = P\cos\pi\gamma$$

$$\pi a\{\delta_*[2\gamma X_0^\circ + (0.25 - \gamma^2)X_1^\circ + \Theta_*[2\gamma X_0 + (0.25 - \gamma^2)X_1^1]\} = M\cos\pi\gamma$$

which allow us to find M and P in terms of the displacement of the punch δ and its angle of rotation Θ, and vice-versa.

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